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UNIVERSAL FAMILIES FOR CONULL FK SPACES

BY

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ABSTRACT. This paper considers the problem of determining a useful family of sequence spaces which is universal for conull FK spaces in the following sense: An FK space is conull if and only if it contains a member of the family. In the equivalent context of weak wedge spaces, an appropriate family of subspaces of boundedness domains m_A of matrices is shown to be universal. Most useful is the fact that the members of this family exhibit unconditional sectional convergence. The latter phenomenon is known for wedge spaces. Another family of spaces which is universal for conull spaces among semiconservative spaces is provided. The spaces are designed to simplify gliding humps arguments. Improvements are thereby obtained for some pseudoconull type theorems of Bennett and Kalton. Finally, it is shown that conull spaces must contain pseudoconull BK algebras.

1. Introduction. A problem of long standing in the theory of conull spaces of summability theory is the determination of a "nice" family of conull spaces which is universal in the following sense: An FK space is conull if and only if it contains a member of this nice family. Solutions to this problem were provided by the author in **[10]** for conservative spaces, and by Zeller in **[14]** for semiconservative matrix convergence domains. In both cases the appropriate conull spaces suitably restricted the oscillation of sequences.

§3 contains some preliminary observations about the weak wedge spaces of Bennett. As an application one obtains the useful result that every weak wedge FK space contains a weak wedge BK space with the coordinate sequences $\{\delta^n\}$ as an unconditional basis. A corresponding theorem for wedge spaces was obtained by Bennett in [**4**, Theorem 1].

§4 introduces a class of spaces which is universal for conull spaces among semiconservative spaces. The members of the family fail even to be semiconservative but exhibit enough of the oscillation properties of conull spaces to be very useful. The spaces are designed with a built-in device for simplifying gliding humps arguments. Improvements are thereby obtained for some pseudoconull type theorems of Bennett in **[4]** and of Bennett and Kalton in **[1]**.

In §5 the pseudoconull property is examined. A space will be called *pseudoconull for separable spaces*, if every separable FK space containing it must be conull. This

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property is useful for obtaining generalizations of the Bounded Consistency Theorem. It is observed that many of the classical pseudoconull spaces are actually algebras. Using a trick of Bennett and Kalton, sufficient conditions are provided for an algebra to be pseudoconull for separable spaces. Also, the principal result of §3 shows that every conull space contains a BK algebra which is pseudoconull for separable spaces.

2. Preliminaries. Let ω denote the space of all real (or complex) sequences. An *FK space* is a subspace of ω which is a locally convex Frechet space on which the coordinate functionals $\{x_n\} \to x_n$ are continuous. A *BK space* is an FK space whose topology is normable. The topology of an FK space is generated by an appropriate sequence of seminorms or a paranorm which ordinarily is a Frechet combination of seminorms. The space ω is an FK space under the topology of coordinatewise convergence.

The following BK spaces will occur in the sequel:

$$bv = \left\{x \in \omega\colon \|x\|_{bv} = |x_1| + \sum_{n=2}^{\infty} |x_n - x_{n-1}| < \infty\right\},$$

$$bs = \left\{x \in \omega\colon \|x\|_{bs} = \sup_n \left|\sum_{k=1}^{n} x_k\right| < \infty\right\},$$

$$cs = \left\{x \in \omega\colon \lim_n \sum_{k=1}^{n} x_k \text{ exists, with norm of } bs\right\},$$

$$m = \left\{x \in \omega\colon \|x\|_\infty = \sup_n |x_n| < \infty\right\},$$

$$c = \left\{x \in \omega\colon \lim_n x_n \text{ exists, with norm of } m\right\},$$

$$c_0 = \left\{x \in \omega\colon \lim_n x_n = 0, \text{ with norm of } m\right\},$$

$$l^p = \left\{x \in \omega\colon \|x\|_p = \left(\sum_n |x_n|^p\right)^{1/p} < \infty\right\}.$$

Let $r = \{r_n\}$ be an increasing sequence of positive integers with $r_1 = 1$. Then

$$\Omega(r) = \left\{x \in \omega\colon \max_{r_n \leq i < j \leq r_{n+1}} |x_i - x_j| \to 0\right\},$$

$$V(r) = \left\{x \in \omega\colon \sum_{k=r_n+1}^{r_{n+1}} |x_k - x_{k-1}| \to 0\right\}.$$

The spaces $\Omega(r)$ and $V(r)$ are BK spaces under

$$\|x\| = \sup_n \max_{r_n \leq i < j \leq r_{n+1}} |x_i - x_j|$$

and

$$\|x\| = \sup_n \sum_{k=r_n+1}^{r_{n+1}} |x_k - x_{k-1}|,$$

respectively.

For $\lambda \in \omega$ let $O(\lambda) = \{x \in \omega: \{x_n/\lambda_n\} \in m\}$.

Let N be the set of positive integers. For each $S \subset N$ let $\chi_S \in \omega$ be the characteristic function of S. For each $n \in N$ let $\delta^n = \chi_{\{n\}}$. Let $1 \in \omega$ denote the constant sequence of ones. ϕ is the span of $\{\delta^n\}$ in ω. For each $n \in N$ let $\psi^n = 1 - \sum_{k=1}^{n-1}\delta^k$.

For $x, y \in \omega$ let $xy \in \omega$ be the sequence $\{x_n y_n\}$. For $x \in \omega$, $E \subset \omega$ let $xE = \{xy: y \in E\}$. Let $E^\alpha = \{y \in \omega: xy \in l^1$ for all $x \in E\}$ and $E^\beta = \{y \in \omega: xy \in cs$ for all $x \in E\}$.

A subset E of ω is said to have a *growth sequence* if there exists $\lambda \in \omega$ such that $E \subset O(\lambda)$.

For $x \in \omega$, $n \in N$ let $P_n x = \sum_{k=1}^{n} x_k \delta^k =$ *nth section* of x. If X is an FK space containing ϕ, let $W_X = \{x \in X: P_n x \to x$ weakly in $X\}$ and $S_X = \{x \in X: P_n x \to x$ in $X\}$.

An FK space $X \supset \phi$ is *semiconservative* if $\lim_n f(P_n 1)$ exists for all $f \in X'$, i.e. if $\{P_n 1\}$ is weakly Cauchy in X. Thus, X is semiconservative if and only if $\{f(\delta^n)\} \in cs$ for all $f \in X'$. An FK space $X \supset \phi + \{1\}$ is *conull* if $1 \in W_X$, i.e. if $\psi^n \to 0$ weakly in X. X is *strongly conull* if $1 \in S_X$.

Following Bennett in **[4]**, an FK space $X \supset \phi$ is a *wedge* (*weak wedge*) *space* if $\delta^n \to 0$ ($\delta^n \to 0$ weakly) in X.

Matrix maps between subsets of ω will occur frequently in what follows. If $A = (a_{ij})$ is an infinite matrix and $x = \{x_j\} \in \omega$, then $Ax \in \omega$ is defined by $(Ax)_i = \sum_j a_{ij} x_j$, assuming the series converges for all i. A is *row-finite* if each of its rows belongs to ϕ.

Let $\omega_A = \{x \in \omega: \sum_j a_{ij} x_j$ converges for all $i\}$. If E is an FK space, let $E_A = \{x \in \omega_A: Ax \in E\}$. It is known that E_A is an FK space under appropriate seminorms.

Particular matrices occuring later are the matrix S defined by $(Sx)_n = \sum_{k=1}^{n} x_k$ and its inverse S^{-1} defined by $(S^{-1}x)_n = x_n - x_{n-1}$ for $n > 1$, $(S^{-1}x)_1 = x_1$. It is easy to see that an FK space X is strongly conull (conull) if and only if $S^{-1}X = X_S$ is wedge (weak wedge).

A thorough treatment of the basic facts about FK spaces and matrix maps may be found in the first several chapters of **[13]**.

3. Observations about weak wedge spaces. Universal families for special classes of conull spaces are known. Proposition 3.1 gives several examples.

3.1. PROPOSITION. (i) (SNYDER **[10]**) *A conservative FK space X is conull if and only if $X \supset \Omega(r)$ for some r.*

(ii) (ZELLER **[14]**) *A semiconservative convergence domain X is conull if and only if $X \supset V(r)$ for some r.*

(iii) (Bennett [4]) *An FK space V is strongly conull if and only if there exists $z \in c_0$ such that*

$$X \supset \left\{x \in \omega\colon \sum |z_n(x_n - x_{n-1})| < \infty\right\}.$$

The $\Omega(r)$ spaces fail as a universal class since each $\Omega(r)$ contains c. The more promising $V(r)$ spaces also fail as shown by Devos in [**6**, 3.15].

3.2. Proposition (Bennett [4]). *The FK space m_A is a weak wedge space if and only if the following conditions are satisfied*:

(i) $\sup_{i,j}|a_{ij}| < \infty$;

(ii) *given $\varepsilon > 0$ and an increasing sequence $\{j_l\}$ of positive integers, there exists L such that* $\sup_i \min\{|a_{i,j_l}|\colon 1 \leqslant l \leqslant L\} < \varepsilon$.

Conditions (i) and (ii) of 3.2 together are equivalent to the requirement that the columns of A converge to 0 weakly in m.

3.3. Theorem. *If X is a weak wedge FK space, then there exists a row-finite matrix A such that*:

(i) *X contains the closure of ϕ in m_A*;

(ii) *m_A is a weak wedge space*;

(iii) *$\{\delta^n\}$ is an unconditional basis for the closure of ϕ in m_A*; *and*

(iv) *the closure of ϕ in $m_A \cap m$ is a weak wedge BK space with $\{\delta^n\}$ as an unconditional basis.*

Proof. Let $\{q_n\}$ be a sequence of seminorms generating the topology of X. For each $n \in N$ and each finite $F \subset N$ let $S_n(F)$ denote all $y \in \omega$ satisfying: y_i is rational for all i, $y_i = 0$ for all $i \notin F$, and there exists $f \in X'$ such that $y_i = f(\delta^i)$ for all $i \in F$ and $|f| \leqslant q_n$. Let $S_n = \bigcup\{S_n(F)\colon F \subset N \text{ finite}\}$. Each S_n is a countable subset of ϕ. Let $A_n = (a^n_{ij})$ be a matrix whose rows are the members of S_n.

Since $\{\delta^k\}$ is bounded in X,

$$\lambda_n = \sup_{i,j}|a^n_{ij}| < \infty$$

for each n. Choose numbers $b_n \neq 0$ such that $\lim_n b_n\lambda_n = 0$. Let $A = (a_{ij})$ be a matrix obtained by intertwining the rows of the matrices b_nA_n. Using 3.2 and a standard compactness argument it is easy to see that m_A is a weak wedge space. On ϕ the topology of X is weaker than the topology of m_A, so X contains the closure of ϕ in m_A.

Now suppose F is a finite of N. If a is any row of A, then $a\chi_F$ is also a row of A. Thus, if $x \in m_A$, then

$$\sup_i\left|\sum_{j\in F} a_{ij}x_j\right| \leqslant \sup_i\left|\sum_{j=1}^{\infty} a_{ij}x_j\right|.$$

It follows that $\{\delta^k\}$ is an unconditional basis for the closure of ϕ in m_A.

Finally, let Z be the closure of ϕ in the weak wedge space $m_A \cap m$. It clearly may be assumed that the matrix A is lower triangular with vanishing diagonal. Let I be the identity matrix. Now $m_A \cap m = m_{A+I} \cap m$ and the latter is a BK space, so Z is a weak wedge BK space. Since Z is contained in the closure of ϕ in m_A, it follows that $\{\delta^k\}$ is an unconditional basis for Z. □

For any $x \in \omega$ define $s^n(x) \in \omega$ by $(s^n(x))_i = x_i$ for $i < n$, x_n for $i \geqslant n$.

It is easy to see that if $x \in \Omega(r)$, then $s^n(x) \to x$ in $\Omega(r)$. By 3.1(i), every conull space $X \supset c$ contains a conull space Y with the property that $s^n(y) \to y$ in Y for all $y \in Y$. The result 3.3 extends this property to arbitrary conull spaces.

3.4. COROLLARY. *Every conull FX space X contains a conull BK space Y such that* $s^n(y) \to y$ *in* Y *for all* $y \in Y$.

PROOF. $S^{-1}(s^n(x)) = P_n(S^{-1}x)$. According to 3.3 the weak wedge space $S^{-1}(X)$ contains a weak wedge BK space Z with basis $\{\delta^n\}$. Then $S(Z)$ is the required conull space. □

3.5. COROLLARY. *An FK space X is conull if and only if there exists a row-finite matrix A satisfying*:

(i) $\sup_{i,j}|\sum_{k=1}^{j} a_{ik}| < \infty$ *and*

(ii) *for each* $\varepsilon > 0$ *and each increasing sequence* $\{j_l\}$ *of positive integers there exists L such that* $\sup_i \min\{|\sum_{k=1}^{j_l} a_{ik}|\colon 1 \leqslant l \leqslant L\} < \varepsilon$

such that X contains the closure of ϕ *in* m_A.

PROOF. According to 3.2, for any row-finite matrix B, m_B is a weak wedge space if and only if the row-finite matrix BS^{-1} satisfies conditions (i) and (ii). Also, $S(m_B) = m_{BS^{-1}}$ and $S\phi \supset \phi$. The result follows from 3.3. □

3.6. COROLLARY. *An FK space X is weak wedge if and only if* S_X *is weak wedge.*

PROOF. Following Garling in [7, pp. 1015–1016], S_X is an FK space with $\{\delta^n\}$ as a basis. An application of 3.3(iii) yields the result. □

The utility of appropriate subspaces of m_A as a universal family of weak wedge spaces is compromised somewhat by the next theorem. It is a large family. However, 3.3 will be very useful in §5.

3.7. THEOREM. *If X is a BK space with basis* $\{\delta^n\}$, *then there is a row-finite matrix A such that X is the closure of* ϕ *in* m_A. *If X has the additional property that* $x \in X$ *whenever* $\{P_n x\}$ *is bounded in X, then there is a row-finite A such that* $X = m_A$.

PROOF. Analogous to the proof of 3.3, for each n let S_n denote all $y \in \omega$ satisfying: y_i rational for all i, $y_i = 0$ for all $i > n$, and there exists $f \in X'$ such that $f(\delta^i) = y_i$ for $i \leqslant n$ and $\|f\| \leqslant 1$. Let A be a matrix whose rows are the members of the countable set $\bigcup_n S_n$. Since the members of X have bounded sections, the topologies of X and m_A agree on ϕ. This proves the first assertion.

Now for $x \in m_A$, $\{P_n x\}$ is bounded in m_A, hence in X. Thus, the additional assumption of the second assertion yields $X = m_A$. □

For $p \geqslant 1$, l^p is a BK space with basis $\{\delta^n\}$ such that $x \in l^p$ whenever $\{P_n x\}$ is bounded in l^p. Therefore, 3.7 is an improvement on a result of Bennett [4, Proposition 9].

4. A universal family for conull spaces. In this section a useful family of concull spaces is developed which is universal for conull spaces among semiconservative spaces. The approach follows the more classical tendency to control the oscillation of sequences.

Let $r = \{r_n\}$ be an increasing sequence of positive integers with $r_1 = 1$. For each n let $v^n \in \omega$ satisfy: $v_i^n = 0$ for $i \leqslant r_{n-1}$, $v_i^n = 1$ for $i \geqslant r_n$, and $v_i^n \leqslant v_{i+1}^n$ for all i. Such a sequence $\{v^n\}$ in ω will be said to satisfy *condition* $(*)$. For each subspace E of ω let

$$V(E, \{v^n\}) = \left\{\sum t_n v^n : t \in E\right\},$$

where $\Sigma t_n v^n$ is the coordinatewise sum.

Note that $t \to \Sigma t_n v^n$ is a one-to-one linear map of E onto $V(E, \{v^n\})$, and that if E is an FK space then $V(E, \{v^n\})$ is an FK space with the seminorms transferred from E. If A is the matrix whose nth column is v^n for each n, then $AE = V(E, \{v^n\})$.

The following lemma is elementary.

4.1. LEMMA. *Assume that* $\{v^n\}$ *satisfies condition* $(*)$.

(i) *For each* k, $(\Sigma t_n v^n)_{r_k} = \Sigma_{n=1}^k t_n$. *Thus, the restriction of* $V(E, \{v^n\})$ *to* $\{r_k\}$ *is just* $S(E)$.

(ii) $V(E, \{v^n\}) \cap m = V(E \cap bs, \{v^n\})$.

(iii) *For each* k, $\Sigma t_n v^n$ *is monotone for* $r_{k-1} \leqslant i \leqslant r_k$.

4.2. PROPOSITION. *The following are equivalent for semiconservative FK spaces X.*

(i) *X is conull.*

(ii) *There exists a bounded sequence* $\{w^n\}$ *in bv such that for each n*, $w_i^n = 1$ *for i large and* $w^n \to 0$ *in X.*

(iii) *There exists an increasing sequence* $r = \{r_n\}$ *of positive integers and a sequence* $v^n \to 0$ *in X such that, for each n*, $v_i^n = 0$ *for* $i \leqslant r_{n-1}$, $v_i^n = 1$ *for* $i \geqslant r_n$, *and* $v_i^n \leqslant v_{i+1}^n$ *for all i.*

PROOF. (i) is equivalent to (ii) by, for instance, [**12**, Theorem 10] (see also [**9**]). Condition (iii) is a minor refinement of (ii) using the usual construction of taking convex combinations of the sequence $\{\psi^n\}$. The latter is weakly null in X if X is conull (see [**3**, Theorem 2], or the proof of 4.5 below). □

4.3. LEMMA. *Let X be an FK space containing* $\phi + \{1\}$ *with seminorms* $\{q_n\}$, *and let the paranorm* $! \cdot !$ *for X be a Frechet combination of* $\{q_n\}$. *Let* $\{v^n\}$ *satisfy condition* $(*)$ *with* $\Sigma !v^n! < \infty$. *If* $E \subset \{!v^n!\}^\alpha$, *then* $V(E, \{v^n\}) \subset X$.

PROOF. If $t \in E$, then

$$\sum !t_n v^n! \leqslant \sum \max\{|t_n|, 1\} !v^n!,$$

so the series $\Sigma t_n v^n$ converges in X. □

4.4. LEMMA. *Assume that* $E \subset \omega$ *has a growth sequence. If X is a conull FK space, then* $X \supset V(E, \{v^n\})$ *for some* $\{v^n\}$ *satisfying condition* $(*)$.

PROOF. Choose $\lambda_n \to \infty$ such that $E \subset O(\{\lambda_n\})$. By 4.2 there exists $\{v^n\}$ satisfying condition $(*)$ such that $\Sigma\lambda_n \| v^n \| < \infty$. The result follows from 4.3. □

4.5. LEMMA. *Assume that E is a weak wedge FK space. If a semiconservative FK space X contains $V(E, \{v^n\})$ for some $\{v^n\}$ satisfying condition $(*)$, then X is conull.*

PROOF. $\delta^n \to 0$ weakly in E so $v^n \to 0$ weakly in $V(E, \{v^n\})$, hence in X. There is a sequence $\{w^n\}$ in the convex hull of $\{v^n\}$ such that $w^n \to 0$ in X, so according to 4.2 the lemma is established. □

4.6. LEMMA. *Let $\{v^n\}$ satisfy condition $(*)$. If $\Sigma t_n v^n \in m$ and converges weakly in the conull FK space X, then $\Sigma t_n v^n \in W_X$.*

PROOF. Suppose $r = \{r_n\}$ is the increasing sequence of positive integers associated with $\{v^n\}$ as in $(*)$. Assume that $r_{n-1} \leqslant k < r_n$. Then if $x = \Sigma t_n v^n$ and $f \in X'$

$$\sum_{i=1}^{k} x_i f(\delta^i) = f\left(\sum_{i=1}^{r_{n-1}} x_i \delta^i\right) + \sum_{i=r_{n-1}+1}^{k} x_i f(\delta^i).$$

Now

$$\left|\sum_{i=r_{n-1}+1}^{k} x_i f(\delta^i)\right| \leqslant \left\|\sum_{i=r_{n-1}+1}^{k} x_i \delta^i\right\|_{bv} \left\|\sum_{i=r_{n-1}+1}^{k} f(\delta^i)\delta^i\right\|_{cs} \to 0$$

as k (and hence n) $\to \infty$, because X is semiconservative. (By 4.1(iii), the first factor on the right is bounded.) Furthermore,

$$f\left(\sum_{i=1}^{r_{n-1}} x_i \delta^i\right) = f\left(\sum_{j=1}^{n-1} t_j v^j\right) - \left(\sum_{j=1}^{n-1} t_j\right) f(\psi^{r_{n-1}+1}) \to f(x),$$

since X is conull, $x \in m$, and since $x = \Sigma t_j v^j$ weakly in X. □

Lemmas 4.4 and 4.5 yield

4.7 THEOREM. *Let E be a weak wedge FK space with a growth sequence. A semiconservative FK space X is conull if and only if there exists $\{v^n\}$ satisfying condition $(*)$ such that $V(E, \{v^n\}) \subset X$.*

It should be noted that $V(c_0, \{v^n\}) + c_0 = \Omega(r)$ so that 3.1(i), the inclusion theorem for conull spaces containing c, is a special case of 4.7.

The following was obtained by Bennett and Kalton in [**1**]. (Actually the proof of 5.2 below yields 4.8 using the well-known fact that $\{\delta^n\}$ is a basis for m with the Mackey topology by l^1.)

4.8. PROPOSITION. *If F is a separable FK space containing m, then F is strongly conull. Equivalently, if F is a separable FK space containing bs, then F is a wedge space.*

The next theorem was obtained in [**1**] by Bennett and Kalton under the assumption that X contains c_0. Another version was obtained in [**11**] by the author under assumptions somewhat weaker than $X \supset c_0$. The result 4.7 easily reduces the general case to 4.8.

4.9. THEOREM. *If X is a conull (weak wedge) FK space and $W_X \cap m \subset F$ ($W_X \cap bs \subset F$) with F a separable FK space, then F is conull (weak wedge).*

PROOF. Let E be a BK space containing m such that $\{\delta^n\}$ is a basis for E. Then E is weak wedge and has a growth sequence. According to 4.7, there exists $\{v^n\}$ satisfying condition (*) such that $V(E, \{v^n\}) \subset X$. By 4.6, $V(E, \{v^n\}) \cap m \subset W_X \cap m \subset F$, so $V(E \cap bs, \{v^n\}) \subset F$.

Let A be the matrix whose nth column is v^n for each n. Then $bs = E \cap bs$ is contained in the separable FK space F_A. By 4.8, F_A is a wedge space, so $v^n = A\delta^n \to 0$ in $A(F_A)$, hence in F.

According to 4.2, it remains to show that F is semiconservative. But 3.4 yields a conull space $Y \subset X$ such that for all $y \in Y$, $s^n(y) = \sum_{k=1}^{n-1} y_k \delta^k + y_n \psi^n \to y$ in Y. It is easy to see that the semiconservative space $Y \cap m$ is contained in $W_X \cap m$, so F is semiconservative.

The corresponding parenthetical result follows from the equality $S(W_X \cap bs) = W_{SX} \cap m$. □

A routine consequence of 4.9 is the following

4.10. COROLLARY. *If X is an FK space containing ϕ and $W_X \cap m \subset F$ with F a separable FK space, then $W_X \cap m \subset W_F$.*

Theorem 16 of [**1**] gave 4.10 under the assumption that $X \supset c_0$.

The next theorem improves a wedge space result of Bennett in [**4**] by replacing m by bs.

4.11. THEOREM. *If X is a wedge (strongly conull) FK space and $S_X \cap bs \subset F$ ($S_X \cap m \subset F$) with F a separable FK space, then F is wedge (strongly conull).*

PROOF. According to the wedge space version of 3.1(iii), it may be assumed that $X \cap bs \subset F$. Using 4.3, there is a subsequence $\{\psi^{r_n}\}$ of $\{\psi^n\}$ such that $V(m, \{\psi^{r_n}\}) \subset SX$, so

$$V(bs, \{\psi^{r_n}\}) = V(m, \{\psi^{r_n}\}) \cap m \subset SF.$$

Let A be the matrix whose nth column is ψ^{r_n} for each n. Then $bs \subset (SF)_A$ so, by 4.8, $(SF)_A$ is a wedge space. Thus, $\psi^{r_n} = A\delta^n \to 0$ in SF, so $\delta^{r_n} \to 0$ in F.

Suppose that F is not a wedge space. There exists an increasing sequence $s = \{s_n\}$ of positive integers such that no subsequence of $\{\delta^{s_n}\}$ converges to 0 in F. For any $E \subset \omega$ let

$$R(E) = \left\{ \{x_{s_i}\} : x \in E, x \text{ vanishes off } s \right\}.$$

It is easy to see that $R(X) \cap bs = R(X \cap bs)$, so $R(X) \cap bs$ is contained in the separable FK space $R(F)$. But $R(X)$ is clearly a wedge space, so by the conclusion of the previous paragraph, a subsequence of $\{\delta^n\}$ converges to 0 in $R(F)$. The latter contradicts the choice of s. □

5. The pseudoconull property and algebras of sequences. Let $\mathscr{R}$ and $\mathscr{S}$ be families of FK spaces. A subset X of ω will be called *pseudo-$\mathscr{R}$ for $\mathscr{S}$* if every member of $\mathscr{S}$ which contains X belongs to $\mathscr{R}$.

For instance, let $\mathscr{R}$ be the conull FK spaces and let $\mathscr{S}$ be the separable FK spaces. Then X is *pseudoconull for separable spaces* if every separable FK space which contants X is conull.

As pointed out by Wilansky in [**13**] it is not known whether pseudoconull for convergence domains implies pseudoconull for separable spaces. (Of course the converse is obvious since convergence domains are separable FK spaces.) This is equivalent to the problem: Does there exist a separable coregular FK space which is pseudoconull for convergence domains?

Theorems 4.9 and 4.11 can be rephrased:

5.1. THEOREM. (i) *If X is a conull (weak wedge) FK space, then $W_X \cap m$ ($W_X \cap bs$) is pseudoconull (pseudoweak wedge) for separable spaces.*

(ii) *If X is a strongly conull (wedge) FK space, then $S_X \cap m$ ($S_X \cap bs$) is pseudo strongly conull (pseudowedge) for separable spaces.*

In particular, for each $r = \{r_n\}$, $\Omega(r) \cap m$ and $V(r) \cap m$ are pseudoconull for separable spaces, since $\Omega(r)$ and $V(r)$ are known to be conull. It was known to Copping in [**5**] that if X is a conull convergence domain containing c, then $X \cap m$ is pseudoconull for convergence domains.

Using a trick of Bennett and Kalton in [**1**], one obtains a sufficient condition for the equivalence of the two pseudoconull properties:

5.2. THEOREM. *Let $X \subset \omega$ be an algebra containing 1. If X is pseudoconull for convergence domains and F is a separable FK space containing X, then $X \subset W_F$; in particular, X is pseudoconull for separable spaces.*

PROOF. Suppose $X \subset c_A$. If $u \in X$ with $u_i \neq 0$ for all i, then

$$X \subset \{1/u_i\} X \subset \{1/u_i\} c_A = c_{Au}$$

where Au is the matrix $(a_{ij}u_j)$. By hypothesis, $1 \in W_{Au} = \{1/u_i\} W_A$, so $u \in W_A$.

It has been shown that $X \subset W_A$ whenever $X \subset c_A$. It follows that X^β is $\sigma(X^\beta, X)$ sequentially complete.

Now assume that X is contained in the separable FK space F. The inclusion mapping clearly has closed graph if X is given the Mackey topology $\tau(X, X^\beta)$. By the Kalton closed graph theorem [**1**, Theorem 13], the inclusion mapping $X \to F$ is continuous. Then [**1**, Proposition 3] yields that the inclusion mapping $(X, \sigma(X, X^\beta)) \to (F, \sigma(F, F'))$ is continuous. For each $x \in X$, $P_n x \to x$ in $\sigma(X, X^\beta)$ so $P_n x \to x$ weakly in F, i.e. $x \in W_F$. □

It is interesting to observe that both $\Omega(r) \cap m$ and $V(r) \cap m$ are algebras under coordinatewise multiplication. Thus, every conull FK space containing c and every conull convergence domain contains an algebra which is pseudoconull for separable spaces. This observation and 5.2 suggest the question: Must every conull FK space contain an algebra containing 1 which is pseudoconull for convergence domains, hence for separable spaces? An affirmative answer is surprisingly easy to obtain using the observations of §3.

5.3. THEOREM. *If X is a conull FX space, then $X \cap m$ contains a semiconservative BK algebra which is pseudoconull for separable spaces.*

PROOF. According to 3.3 the weak wedge FK space $S^{-1}X$ contains a weak wedge BK space Y with $\{\delta^n\}$ as an unconditional basis. Now SY is conull, so by 5.1 the semiconservative BK space $(SY) \cap m \subset X$ is pseudoconull for separable spaces. Finally, if $x \in (SY) \cap m$, then $\{x_n + x_{n-1}\} \in m$. It follows that $S^{-1}(\{x_n^2\}) \in Y$, because $\{\delta^n\}$ is an unconditional basis for Y. Therefore, $\{x_n^2\} \in (SY) \cap m$, so $(SY) \cap m$ is an algebra. □

Little is known about the pseudoconull property. It would be most interesting if pseudoconull spaces had nice intersection properties under suitable assumptions. Much of the bounded consistency theory is related to intersections of appropriate pseudoconull spaces. See, for example, [**11**, Theorem 2]. A useful sufficient condition for the property of pseudoconull for convergence domains is given in [**11**, Lemma 1].

To illustrate the lack of intersection properties, there exists a conull space X and a space Y which is pseudocoercive for separable spaces (i.e. every separable space containing Y must contain m) such that $X \cap Y = c$:

5.4. EXAMPLE. There exists a closed subspace S of m such that $c_0 \cap S = \{0\}$ and $c_0 + S$ is dense in m (see [**8**, p. 185].) Let $u \in \omega$ be the sequence $\{n^2\}$. Let $Y = uS + c$. Take X to be the conull space $O(\{n\})$.

Now Y is pseudocoercive for separable spaces, for suppose $uS + c \subset F$ with F a separable FK space. Then $S + \{1/u_n\}c$ is contained in the separable FK space $\{1/u_n\}F$. It is clear that $S + \{1/u_n\}c$ is dense in m. According to [**2**, Theorem 3], $m \subset \{1/u_n\}F$ so $m \subset um \subset F$.

But $X \cap Y = c$. To see this, suppose $us + a \in O(\{n\})$ where $s \in S$, $a \in c$. Then $s \in \{1/n^2\}O(\{n\}) \subset c_0$. Since $S \cap c_0 = \{0\}$, $s = 0$ so $us + a \in c$. □

Finally, it should be observed that the separabiltiy hypothesis in 4.8 can be weakened somewhat, as in [**2**], to the requirement that the space contains no closed subspace isomorphic to m. Thus, it can be shown easily that the same change can be made in 4.9–4.11 and 5.1. The author is unable to determine if 5.3 can be altered similarly.

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